Chapter 7

Basic Methods of Integration

Learning the art of integration requires practice.

In this chapter, we first collect in a more systematic way some of the integration formulas derived in Chapters 4–6. We then present the two most important general techniques: integration by substitution and integration by parts. As the techniques for evaluating integrals are developed, you will see that integration is a more subtle process than differentiation and that it takes practice to learn which method should be used in a given problem.

7.1 Calculating Integrals

The rules for differentiating the trigonometric and exponential functions lead to new integration formulas.

In this section, we review the basic integration formulas learned in Chapter 4, and we summarize the integration rules for trigonometric and exponential functions developed in Chapters 5 and 6.

Given a function $f(x)$, $\int f(x)\,dx$ denotes the general antiderivative of f, also called the indefinite integral. Thus

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is a constant. Therefore,

$$\frac{d}{dx}\int f(x)\,dx = f(x).$$

The definite integral is obtained via the fundamental theorem of calculus by evaluating the indefinite integral at the two limits and subtracting. Thus:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

We recall the following general rules for antiderivatives (see Section 2.5), which may be deduced from the corresponding differentiation rules. To check the sum rule, for instance, we must see if

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = f(x) + g(x).$$

But this is true by the sum rule for derivatives.

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Sum and Constant Multiple Rules for Antiderivatives

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx;$$
$$\int cf(x)\,dx = c\int f(x)\,dx.$$

The antiderivative rule for powers is given as follows:

Power Rule for Antiderivatives

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & n \neq -1, \\ \ln|x| + C, & n = -1. \end{cases}$$

The power rule for integer n was introduced in Section 2.5, and was extended in Section 6.3 to cover the case $n=-1$ and then to all real numbers n, rational or irrational.

Example 1 Calculate (a) $\int\left(3x^{2/3}+\frac{8}{x}\right)dx$; (b) $\int\left(\frac{x^3+8x+3}{x}\right)dx$; (c) $\int(x^\pi + x^3)\,dx$.

Solution (a) By the sum and constant multiple rules,

$$\int\left(3x^{2/3}+\frac{8}{x}\right)dx = 3\int x^{2/3}\,dx + 8\int\frac{1}{x}\,dx.$$

By the power rule, this becomes

$$3\cdot\frac{x^{5/3}}{5/3} + 8\ln|x| + C = \frac{9}{5}x^{5/3} + 8\ln|x| + C.$$

(b) $$\int\frac{x^3+8x+3}{x}\,dx = \int\left(x^2+8+\frac{3}{x}\right)dx = \frac{x^3}{3} + 8x + 3\ln|x| + C.$$

(c) $$\int(x^\pi + x^3)\,dx = \frac{x^{\pi+1}}{\pi+1} + \frac{x^4}{4} + C. \blacktriangle$$

Applying the fundamental theorem to the power rule, we obtain the rule for definite integrals of powers:

Definite Integral of a Power

$$\int_a^b x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1}-a^{n+1}}{n+1} \qquad \text{for } n \text{ real}, \quad n \neq -1.$$

If $n=-2,-3,-4,\ldots,$ a and b must have the same sign. If n is not an integer, a and b must be positive (or zero if $n>0$).

$$\int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b = \ln|b| - \ln|a| = \ln\left(\frac{b}{a}\right).$$

Again a and b must have the same sign.

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The extra conditions on a and b are imposed because the integrand must be defined and continuous on the domain of integration; otherwise the fundamental theorem does not apply. (See Exercise 46.)

Example 2 Evaluate (a) $\int_0^1 (x^4 - 3\sqrt{x})\,dx$; (b) $\int_1^2 \left(\sqrt{x} + \frac{2}{x}\right) dx$; (c) $\int_{1/2}^1 \left(\frac{x^4 + x^6 + 1}{x^2}\right) dx$.

Solution (a)

$$\int_0^1 (x^4 - 3\sqrt{x})\,dx = \int (x^4 - 3\sqrt{x})\,dx\Big|_0^1 = \frac{x^5}{5} - 3\cdot\frac{x^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{1}{5} - 2 = -\frac{9}{5}.$$

(b)

$$\int_1^2 \left(\sqrt{x} + \frac{2}{x}\right) dx = \left(\frac{x^{3/2}}{3/2} + 2\ln|x|\right)\Big|_1^2$$

$$= \frac{2}{3}2^{3/2} + 2\ln 2 - \left(\frac{2}{3} + 0\right) = \frac{4\sqrt{2} - 2}{3} + 2\ln 2.$$

(c)

$$\int_{1/2}^1 \left(\frac{x^4 + x^6 + 1}{x^2}\right) dx = \int_{1/2}^1 \left(x^2 + x^4 + \frac{1}{x^2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{x^5}{5} - \frac{1}{x}\right)\Big|_{1/2}^1$$

$$= \left(\frac{1}{3} + \frac{1}{5} - 1\right) - \left(\frac{1}{3\cdot 8} + \frac{1}{5\cdot 32} - 2\right)$$

$$= \frac{713}{480}. \blacktriangle$$

In the following box, we recall some general properties satisfied by the definite integral. These properties were discussed in Chapter 4.

Properties of the Definite Integral

1. *Inequality rule*: If $f(x) \leqslant g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

2. *Sum rule*:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

3. *Constant multiple rule*:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \qquad c \text{ a constant.}$$

4. *Endpoint additivity rule*:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \qquad a < b < c.$$

5. *Wrong-way integrals*:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

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If we consider the integral as the area under the graph, then the endpoint additivity rule is just the principle of addition of areas (see Fig. 7.1.1).

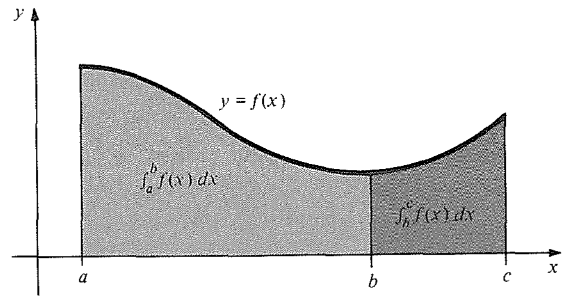

Figure 7.1.1. The area of the entire figure is $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$, which is the sum of the areas of the two subfigures.

Example 3 Let

$$f(t) = \begin{cases} \frac{1}{2} & 0 \leqslant t < \frac{1}{2}, \\ t, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Draw a graph of f and evaluate $\int_0^1 f(t)\,dt$.

Solution The graph of f is drawn in Fig. 7.1.2. To evaluate the integral, we apply the endpoint additivity rule with $a = 0$, $b = \frac{1}{2}$, and $c = 1$:

$$\int_0^1 f(t)\,dt = \int_0^{1/2} f(t)\,dt + \int_{1/2}^1 f(t)\,dt = \int_0^{1/2} \frac{1}{2}\,dt + \int_{1/2}^1 t\,dt$$

$$= \frac{1}{2}t\Big|_0^{1/2} + \frac{1}{2}t^2\Big|_{1/2}^1 = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}. \blacktriangle$$

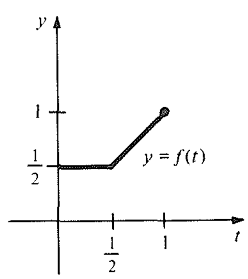

Figure 7.1.2. The integral of f on $[0, 1]$ is the sum of its integrals on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Let us recall that the alternative form of the fundamental theorem of calculus states that if f is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example 4 Find $\displaystyle\frac{d}{dt}\int_0^{t^2} \sqrt{1 + 2s^3}\,ds$.

Solution We write $g(t) = \int_0^{t^2} \sqrt{1 + 2s^3}\,ds$ as $f(t^2)$, where $f(u) = \int_0^u \sqrt{1 + 2s^3}\,ds$. By the fundamental theorem (alternative version), $f'(u) = \sqrt{1 + 2u^3}$; by the chain rule, $g'(t) = f'(t^2)[d(t^2)/dt] = \sqrt{1 + 2t^6} \cdot 2t$. $\blacktriangle$

As we developed the calculus of the trigonometric and exponential functions, we obtained formulas for the antiderivatives of certain of these functions. For convenience, we summarize those formulas. Here are the formulas from Chapter 5:

Trigonometric Formulas

1. $\int \cos\theta\,d\theta = \sin\theta + C$
2. $\int \sin\theta\,d\theta = -\cos\theta + C$
3. $\int \sec^2\theta\,d\theta = \tan\theta + C$
4. $\int \csc^2\theta\,d\theta = -\cot\theta + C$
5. $\int \tan\theta\sec\theta\,d\theta = \sec\theta + C$
6. $\int \cot\theta\csc\theta\,d\theta = -\csc\theta + C$

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Inverse Trigonometric Formulas

1. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C, \qquad -1 < x < 1.$
2. $\displaystyle\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1}x + C, \qquad -1 < x < 1.$
3. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + C, \qquad -\infty < x < \infty.$
4. $\displaystyle\int \frac{-dx}{1+x^2} = \cot^{-1}x + C, \qquad -\infty < x < \infty.$
5. $\displaystyle\int \frac{dx}{\sqrt{x^2(x^2-1)}} = \sec^{-1}x + C, \qquad -\infty < x < -1 \text{ or } 1 < x < \infty.$
6. $\displaystyle\int \frac{-dx}{\sqrt{x^2(x^2-1)}} = \csc^{-1}x + C, \qquad -\infty < x < -1 \text{ or } 1 < x < \infty.$

By combining the fundamental theorem of calculus with these formulas and the ones in the tables on the endpapers of this book, we can compute many definite integrals.

Example 5 Evaluate (a) $\int_0^{\pi}(x^4 + 2x + \sin x)\,dx$; (b) $\int_0^{\pi/6}\cos 3x\,dx$; (c) $\int_{-1/2}^{1/2}\frac{dy}{\sqrt{1-y^2}}$.

Solution (a) We begin by calculating the indefinite integral, using the sum and constant multiple rules, the power rule, and the fact that the antiderivative of $\sin x$ is $-\cos x + C$:

$$\begin{aligned}\int(x^4 + 2x + \sin x)\,dx &= \int x^4\,dx + 2\int x\,dx + \int \sin x\,dx\\ &= x^5/5 + x^2 - \cos x + C.\end{aligned}$$

The fundamental theorem then gives

$$\begin{aligned}&\int_0^{\pi}(x^4 + 2x + \sin x)\,dx\\ &\quad = \left(\frac{x^5}{5} + x^2 - \cos x\right)\Bigg|_0^{\pi} = \frac{\pi^5}{5} + \pi^2 - \cos\pi - (0 + 0 - \cos 0)\\ &\quad = \frac{\pi^5}{5} + \pi^2 + 1 + 1 = 2 + \pi^2 + \frac{\pi^5}{5} \approx 73.07.\end{aligned}$$

(b) An antiderivative of $\cos 3x$ is, by guesswork, $\frac{1}{3}\sin 3x$. Thus

$$\int_0^{\pi/6}\cos 3x\,dx = \frac{1}{3}\sin 3x\Big|_0^{\pi/6} = \frac{1}{3}\sin\frac{\pi}{2} = \frac{1}{3}.$$

(c) From the preceding box, we have

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1}y + C,$$

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and so by the fundamental theorem,

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1}y\Big|_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}. \blacktriangle$$

The following box summarizes the antidifferentiation formulas obtained in Chapter 6.

Exponential and Logarithm

$$\int e^x\,dx = e^x + C,$$
$$\int b^x\,dx = \frac{b^x}{\ln b} + C,$$
$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Example 6 Find (a) $\int_{-1}^{1} 2^x\,dx$; (b) $\int_0^1 (3e^x + 2\sqrt{x})\,dx$; (c) $\int_0^1 2^{2y}\,dy$.

Solution (a) $\displaystyle\int_{-1}^{1} 2^x\,dx = \frac{2^x}{\ln 2}\Big|_{-1}^{1} = \frac{2}{\ln 2} - \frac{2^{-1}}{\ln 2} = \frac{3}{2\ln 2} \approx 2.164.$

(b)
$$\int_0^1 (3e^x + 2\sqrt{x})\,dx = 3\int_0^1 e^x\,dx + 2\int_0^1 x^{1/2}\,dx$$
$$= 3e^x\big|_0^1 + 2\left(\frac{x^{3/2}}{3/2}\right)\Big|_0^1$$
$$= 3(e^1 - e^0) + \frac{4}{3}(1^{3/2} - 0^{3/2})$$
$$= 3e - 3 + \frac{4}{3} = 3e - \frac{5}{3} \approx 6.488.$$

(c) By a law of exponents, $2^{2y} = (2^2)^y = 4^y$. Thus,

$$\int_0^1 2^{2y}\,dy = \int_0^1 4^y\,dy = \frac{4^y}{\ln 4}\Big|_0^1 = \frac{1}{\ln 4}(4-1) = \frac{3}{2\ln 2}.$$

Example 7 (a) Differentiate $x\ln x$. (b) Find $\int \ln x\,dx$. (c) Find $\int_2^5 \ln x\,dx$.

Solution (a) By the product rule for derivatives,

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1.$$

(b) From (a), $\int(\ln x + 1)\,dx = x\ln x + C$. Hence,

$$\int \ln x\,dx = x\ln x - x + C.$$

(c)
$$\int_2^5 \ln x\,dx = (x\ln x - x)\big|_2^5 = (5\ln 5 - 5) - (2\ln 2 - 2)$$
$$= 5\ln 5 - 2\ln 2 - 3. \blacktriangle$$

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Finally we recall by means of a few examples how integrals can be used to solve area and rate problems.

Example 8 (a) Find the area between the x axis, the curve $y = 1/x$, and the lines $x = -e^3$ and $x = -e$.
(b) Find the area between the graphs of $\cos x$ and $\sin x$ on $[0, \pi/4]$.

Solution (a) For $-e^3 \leqslant x \leqslant -e$, we notice that $1/x$ is negative. Therefore the graph of $1/x$ lies below the x axis (the graph of $y = 0$), and the area is

$$\int_{-e^3}^{-e}\left(0 - \frac{1}{x}\right)dx = -\ln|x|\Big|_{-e^3}^{-e} = -(\ln e - \ln e^3) = -(1-3) = 2.$$

See Fig. 7.1.3.

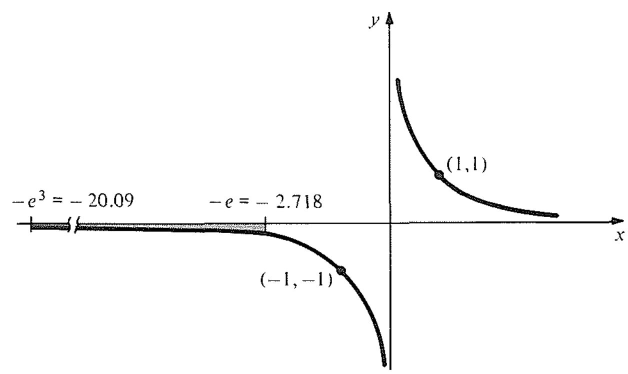

Figure 7.1.3. Find the shaded area.

(b) Since $0 \leqslant \sin x \leqslant \cos x$ for x in $[0, \pi/4]$ (see Fig. 7.1.4), the formula

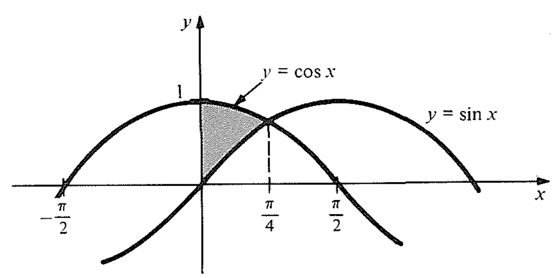

Figure 7.1.4. Find the area of the shaded region.

for the area between two graphs (see Section 4.6) gives

$$\int_0^{\pi/4}(\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1. \blacktriangle$$

Example 9 Water flows into a tank at the rate of $2t + 3$ liters per minute, where t is the time measured in *hours* after noon. If the tank is empty at noon and has a capacity of 1000 liters, when will it be full?

Solution First we should express everything in terms of the same unit of time. Choosing hours, we convert the rate of $2t + 3$ liters per minute to $60(2t + 3) = 120t + 180$ liters per hour. The total amount of water in the tank at time T hours past noon is the integral

$$\int_0^T(120t + 180)\,dt = \frac{120}{2}(T^2 - 0^2) + 180(T - 0) = 60T^2 + 180T.$$

The tank is full when $60T^2 + 180T = 1000$. Solving for T by the quadratic formula, we find $T \approx 2.849$ hours past noon, so the tank is full at 2:51 P.M. $\blacktriangle$

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Example 10 Let $P(t)$ denote the population of bacteria in a certain colony at time t. Suppose that $P(0) = 100$ and that P is increasing at a rate of $20e^{3t}$ bacteria per day at time t. How many bacteria are there after 50 days?

Solution We are given $P'(t) = 20e^{3t}$ and $P(0) = 100$. Taking the antiderivative of $P'(t)$ gives $P(t) = \frac{20}{3}e^{3t} + C$. Substituting $P(0) = 100$ gives $C = 100 - \frac{20}{3}$. Hence $P(t) = 100 + \frac{20}{3}(e^{3t} - 1)$, and $P(50) = 100 + \frac{20}{3}(e^{150} - 1) \approx 9.2 \times 10^{65}$ bacteria. (This exceeds the number of atoms in the universe, so growth cannot go on at such a rate and our model for bacterial growth must become invalid.) ▲

Exercises for Section 7.1

Evaluate the indefinite integrals in Exercises 1–8.

1. $\int (3x^2 + 2x + x^{-3})\,dx$
2. $\int (8x^2 + 3x^{-4} + x^{-8})\,dx$
3. $\int (e^x + 2x)\,dx$
4. $\int (e^{-2x} - 8x^2)\,dx$
5. $\int (\sin 2x + 3x)\,dx$
6. $\int (\cos 3x - 2x + 1)\,dx$
7. $\int (e^{-x} + 2\cos x + 5x^2)\,dx$
8. $\int (e^{3x} - 8\sin 2x + x^{-4})\,dx$

Evaluate the definite integrals in Exercises 9–34.

9. $\int_{-2}^{2} (x^8 + 2x^2 - 1)\,dx$
10. $\int_{-2}^{2} (x^{16} + x^9)\,dx$
11. $\int_{3}^{6} (1 - y + y^2)\,dy$
12. $\int_{0}^{a} (6x^2 + 3x + 2)\,dx$
13. $\int_{16}^{81} \sqrt[4]{s}\,ds$
14. $\int_{1}^{81} s\sqrt[4]{s}\,ds$
15. $\int_{-4}^{-3} \frac{1}{r^2}\,dr$
16. $\int_{1}^{3} \left(\frac{1}{x^2} + \frac{1}{x^3}\right) dx$
17. $\int_{-\pi}^{\pi} \cos x\,dx$
18. $\int_{0}^{\pi/2} \sin 5x\,dx$
19. $\int_{0}^{\pi} (3\sin\theta + 4\cos\theta)\,d\theta$
20. $\int_{0}^{\pi/2} (3\sin 4x + 4\cos 3x)\,dx$
21. $\int_{0}^{1} \frac{3}{x^2 + 1}\,dx$
22. $\int_{0}^{1} \frac{ds}{1 + s^2}$
23. $\int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2 - 1}}$
24. $\int_{0}^{\sqrt{2}/2} (4 - 4s^2)^{-1/2}\,ds$
25. $\int_{0}^{\pi/4} \sec^2 x\,dx$
26. $\int_{0}^{\pi/4} \left(e^x - \frac{3}{\cos^2 x}\right) dx$
27. $\int_{1}^{2} (e^{3x} + x^{2/3})\,dx$
28. $\int_{-1}^{1} e^{-4x}\,dx$
29. $\int_{1}^{5} \frac{1}{t}\,dt$
30. $\int_{3/2}^{2} \frac{dx}{20x}$
31. $\int_{1}^{2} \frac{1 + 2x + 3x^2 + 4x^3}{x^4}\,dx$
32. $\int_{1}^{2} \left[x + \frac{1}{x}\right]^2 dx$
33. $\int_{-200}^{200} (90x^{21} - 80x^{33} + 5580x^{97} + 1)\,dx$
34. $\int_{-243.8}^{243.8} (65x^{73} + 48x^{29} - 3x^{13} + 15x^5 - 2x)\,dx.$
35. Check the formula
$$\int x\sqrt{1 + x}\,dx = \frac{2}{15}(3x - 2)(1 + x)^{3/2} + C$$
and evaluate $\int_{0}^{3} x\sqrt{1 + x}\,dx$.
36. (a) Check the integral
$$\int \frac{1}{x\sqrt{x - 1}}\,dx = 2\tan^{-1}\sqrt{x - 1} + C.$$
(b) Evaluate $\int_{2}^{4} (1/x\sqrt{x - 1})\,dx$.
37. (a) Verify that $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$.
(b) Evaluate $\int_{1}^{e} (2xe^{x^2} + 3\ln x)\,dx$ (see Example 7).

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38. (a) Verify the formula

$$\int\left[\frac{\sqrt{x^2-1}}{x}\right]dx=\sqrt{x^2-1}-\sec^{-1}x+C.$$

(b) Evaluate $\int_1^{3/2}[\sqrt{x^2-1}/x]\,dx$.

39. Suppose that $\int_0^2 f(t)\,dt=5$, $\int_2^5 f(t)\,dt=6$, and $\int_0^7 f(t)\,dt=3$. Find (a) $\int_0^5 f(t)\,dt$ and (b) $\int_5^7 f(t)\,dt$. (c) Show that $f(t)<0$ for some t in $(5,7)$.

40. Find $\int_1^3[4f(s)+3/\sqrt[3]{s}\,]\,ds$, where $\int_1^3 f(s)\,ds=6$.

41. Find $\frac{d}{dt}\int_{t^2}^4\sqrt{e^x+\sin 5x^2}\,dx$.

42. Compute $\frac{d}{d\alpha}\int_0^{\alpha^2}(\sin^2 t+e^{\cos t})^{3/2}\,dt$.

43. Let

$$f(t)=\begin{cases}2 & -1\leqslant t<0,\\ t & 0\leqslant t\leqslant 2,\\ -1 & 2<t\leqslant 3.\end{cases}$$

Compute $\int_{-1}^3 f(t)\,dt$.

44. Let

$$h(x)=\begin{cases}x & 0\leqslant x<\frac{3}{4},\\ \frac{3}{4} & \frac{3}{4}\leqslant x<1.\end{cases}$$

Compute $\int_0^1 h(x)\,dx$.

45. Let $f(x)=\sin x$,

$$g(x)=\begin{cases}1 & -\pi\leqslant x\leqslant 2,\\ 2 & 2<x\leqslant\pi,\end{cases}$$

and $h(x)=1/x^2$. Find:

(a) $\int_{-\pi/2}^{\pi/2} f(x)g(x)\,dx$; (b) $\int_1^3 g(x)h(x)\,dx$;

(c) $\int_{\pi/2}^x f(t)g(t)\,dt$, for x in $(0,\pi]$. Draw a graph of this function of x.

46. We have $1/x^4>0$ for all x. On the other hand, $\int(dx/x^4)=\int x^{-4}\,dx=(x^{-3}/-3)+C$, so

$$\int_{-1}^1\frac{dx}{x^4}=\frac{1^{-3}-(-1)^{-3}}{-3}=\frac{1+1}{-3}=-\frac{2}{3}.$$

How can a positive function have a negative integral?

Find the area under the graph of each of the functions in Exercises 47–50 on the stated interval.

47. $\dfrac{x^3+1}{x^2+1}$ on $[0,2]$. [*Hint:* Divide.]

48. $\dfrac{1}{x^2+1}$ on $[0,2]$.

49. $\dfrac{x^2+2}{\sqrt{x}}$ on $[1,4]$.

50. $\sin x-\cos 2x$ on $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$.

51. Find the area under the graph of $y=e^{2x}$ between $x=0$ and $x=1$.

52. A region containing the origin is cut out by the curves $y=1/\sqrt{x}$, $y=-1/\sqrt{x}$, $y=1/\sqrt{-x}$, and $y=-1/\sqrt{-x}$ and the lines $x=\pm4$, $y=\pm4$; see Fig. 7.1.5. Find the area of this region.

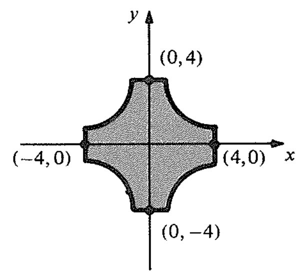

Figure 7.1.5. Find the area of the shaded region.

53. Find the area of the shaded region in Fig. 7.1.6.

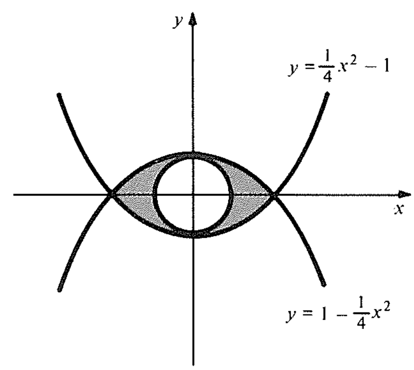

Figure 7.1.6. Find the area of the "retina."

54. Find the area of the shaded "flower" in Fig. 7.1.7.

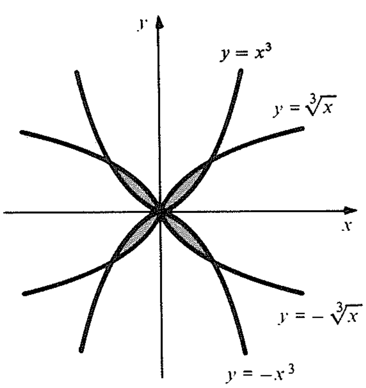

Figure 7.1.7. Find the shaded area.

55. Illustrate in terms of areas the fact that

$$\int_0^{n\pi}\sin x\,dx=\begin{cases}2, & \text{if } n \text{ is an odd positive integer;}\\ 0, & \text{if } n \text{ is an even positive integer.}\end{cases}$$

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56. Find the area of the shaded region in Fig. 7.1.8.

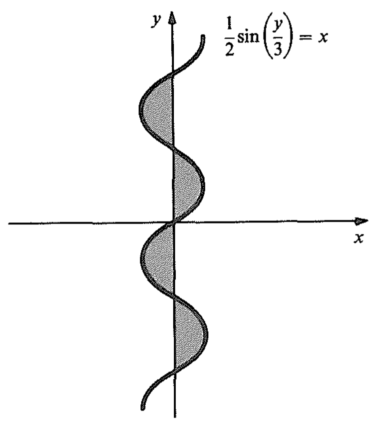

Figure 7.1.8. Find the area of the shaded region.

57. Assuming without proof that

$$\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx \quad \text{(see Fig. 7.1.9)},$$

find $\int_0^{\pi/2}\sin^2 x\,dx$. (*Hint*: $\sin^2 x + \cos^2 x = 1$.)

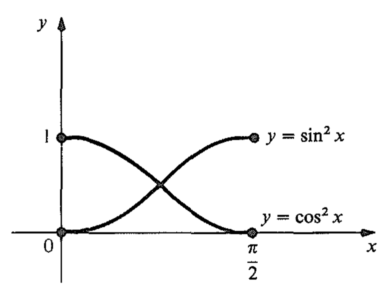

Figure 7.1.9. The areas under the graphs of $\sin^2 x$ and $\cos^2 x$ on $[0, \pi/2]$ are equal.

58. Find:
   (a) $\int \cos 2x\,dx$;
   (b) $\int (\cos^2 x - \sin^2 x)\,dx$;
   (c) $\int (\cos^2 x + \sin^2 x)\,dx$;
   (d) $\int \cos^2 x\,dx$ (use parts (b) and (c));
   (e) $\int_0^{\pi/2}\cos^2 x\,dx$ and $\int_0^{\pi/2}\sin^2 x\,dx$ (compare with Exercise 57).

59. (a) Show that $\int \sin t \cos t\,dt = \frac{1}{2}\sin^2 t + C$.
   (b) Using the identity $\sin 2t = 2\sin t\cos t$, show that $\int \sin t\cos t\,dt = -\frac{1}{4}\cos 2t + C$.
   (c) Use each of parts (a) and (b) to compute $\int_{\pi/6}^{\pi/4}\sin t\cos t\,dt$. Compare your answers.

60. Find the area of the shaded region in Fig. 7.1.10.

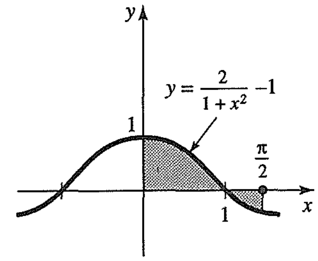

Figure 7.1.10. Find the shaded area.

61. Show that the area under the graph of $f(x) = 1/(1+x^2)$ on $[a, b]$ is less than π, no matter what the values of a and b may be.

62. Show: the area under the graph of $1/(x^2 + x^6)$ between $x = 2$ and $x = 3$ is smaller than $\frac{1}{68}$.

63. A particle starts at the origin and has velocity $v(t) = 7 + 4t^3 + 6\sin(\pi t)$ centimeters per second after t seconds. Find the distance travelled in 200 seconds.

64. The sales of a clothing company t days after January 1 are given by $S(t) = 260e^{(0.1)t}$ dollars per day.
   (a) Set up a definite integral which gives the accumulated sales on $0 \leqslant t \leqslant 10$.
   (b) Find the accumulated sales for the first 10 days.
   (c) How many days must pass before sales exceed \$900 per day?

65. Each unit in a four-plex rents for \$230/month. The owner will trade the property in five years. He wants to know the *capital value* of the property over a five-year period for continuous interest of 8.25%, that is, the amount he could borrow *now* at 8.25% continuous interest, to be paid back by the rents over the next five years. This amount A is given by $A = \int_0^T Re^{-kt}\,dt$, where R = annual rents, k = annual continuous interest rate, T = period in years.
   (a) Verify that $A = (R/k)(1 - e^{-kT})$.
   (b) Find A for the four-plex problem.

66. The *strain energy* V_e for a simply supported uniform beam with a load P at its center is

$$V_e = \frac{1}{EI}\int_0^{l/2}\left(\frac{Px}{2}\right)^2 dx.$$

The *flexural rigidity* EI and the *bar length* l are constants, $EI \neq 0$ and $l > 0$. Find V_e.

67. A manufacturer determines by curve-fitting methods that its marginal revenue is given by $R'(t) = 1000e^{t/2}$ and its marginal cost by $C'(t) = 1000 - 2t$, t days after January 1. The revenue and cost are in dollars.
   (a) Suppose $R(0) = 0$, $C(0) = 0$. Find, by means of integration, formulas for $R(t)$ and $C(t)$.
   (b) The total profit is $P = R - C$. Find the total profit for the first seven days.

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68. The probability P that a capacitor manufactured by an electronics company will last between three and five years with normal use is given approximately by $P = \int_3^5 (22.05)t^{-3}\,dt$.

(a) Find the probability P.

(b) Verify that $\int_3^7 (22.05)t^{-3}\,dt = 1$, which says that all capacitors have expected life between three and seven years.

69. Using the identity $\frac{1}{t} - \frac{1}{t+1} = \frac{1}{t(t+1)}$, find

$$\int_1^e \frac{dt}{t(t+1)}.$$

★70. Compute $\int \frac{dt}{t^2(t+1)}$ by writing

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{(t+1)}$$

for suitable constants A, B, C.

7.2 Integration by Substitution

Integrating the chain rule leads to the method of substitution.

The method of integration by substitution is based on the chain rule for differentiation. If F and g are differentiable functions, the chain rule tells us that $(F \circ g)'(x) = F'(g(x))g'(x)$; that is, $F(g(x))$ is an antiderivative of $F'(g(x))g'(x)$. In indefinite integral notation, we have

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

As in differentiation, it is convenient to introduce an intermediate variable $u = g(x)$; then the preceding formula becomes

$$\int F'(u)\frac{du}{dx}\,dx = F(u) + C.$$

If we write $f(u)$ for $F'(u)$, so that $\int f(u)\,du = F(u) + C$, we obtain the formula

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du. \tag{1}$$

This formula is easy to remember, since one may "cancel the dx's."

To apply the method of substitution one must find in a given integrand an expression $u = g(x)$ whose derivative $du/dx = g'(x)$ also occurs in the integrand.

Example 1 Find $\int 2x\sqrt{x^2+1}\,dx$ and check the answer by differentiation.

Solution None of the rules in Section 7.1 apply to this integral, so we try integration by substitution. Noticing that $2x$, the derivative of $x^2 + 1$, occurs in the integrand, we are led to write $u = x^2 + 1$; then we have

$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\cdot 2x\,dx = \int \sqrt{u}\left(\frac{du}{dx}\right)dx.$$

By formula (1), the last integral equals $\int\sqrt{u}\,du = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C$. At this point we substitute $x^2 + 1$ for u, which gives

$$\int 2x\sqrt{x^2+1}\,dx = \tfrac{2}{3}(x^2+1)^{3/2} + C.$$

Checking our answer by differentiating has educational as well as insur-

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ance value, since it will show how the chain rule produces the integrand we started with:

$$\frac{d}{dx}\left[\frac{2}{3}(x^2+1)^{3/2}+C\right]=\frac{2}{3}\cdot\frac{3}{2}(x^2+1)^{1/2}\frac{d}{dx}(x^2+1)=\left[\sqrt{x^2+1}\,\right]2x,$$

as it should be. ▲

Sometimes the derivative of the intermediate variable is "hidden" in the integrand. If we are clever, however, we can still use the method of substitution, as the next example shows.

Example 2 Find $\int \cos^2 x \sin x\, dx$.

Solution We are tempted to make the substitution $u=\cos x$, but du/dx is then $-\sin x$ rather than $\sin x$. No matter—we can rewrite the integral as

$$\int(-\cos^2 x)(-\sin x)\,dx.$$

Setting $u=\cos x$, we have

$$\int -u^2\frac{du}{dx}\,dx=\int -u^2\,du=-\frac{u^3}{3}+C,$$

so

$$\int \cos^2 x\sin x\,dx=-\tfrac{1}{3}\cos^3 x+C.$$

You may check this by differentiating. ▲

Example 3 Find $\int \frac{e^x}{1+e^{2x}}\,dx$.

Solution We cannot just let $u=1+e^{2x}$, because $du/dx=2e^{2x}\neq e^x$; but we may recognize that $e^{2x}=(e^x)^2$ and remember that the derivative of e^x is e^x. Making the substitution $u=e^x$ and $du/dx=e^x$, we have

$$\begin{aligned}\int\frac{e^x}{1+e^{2x}}\,dx&=\int\frac{1}{1+(e^x)^2}\cdot e^x\,dx\\&=\int\frac{1}{1+u^2}\cdot\frac{du}{dx}\cdot dx=\int\frac{1}{1+u^2}\,du\\&=\tan^{-1}u+C=\tan^{-1}(e^x)+C.\end{aligned}$$

Again you should check this by differentiation. ▲

We may summarize the method of substitution as developed so far (see Fig. 7.2.1).

Integration by Substitution

To integrate a function which involves an intermediate variable u and its derivative du/dx, write the integrand in the form $f(u)(du/dx)$, incorporating constant factors as required in $f(u)$. Then apply the formula

$$\int f(u)\frac{du}{dx}\,dx=\int f(u)\,du.$$

Finally, evaluate $\int f(u)\,du$ if you can; then substitute for u its expression in terms of x.

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$$\int \underbrace{(\text{expression in } u)}_{\boxed{\text{u appears here as a function}}} \cdot \underbrace{(\text{derivative of } u)}_{\boxed{\text{the derivative of u appears as a factor}}}\, dx = \int (\text{expression in } u)\, du$$

Figure 7.2.1. How to spot u in a substitution problem.

Example 4 Find (a) $\int x^2 \sin(x^3)\,dx$, (b) $\int \sin 2x\,dx$.

Solution (a) We observe that the factor x^2 is, apart from a factor of 3, the derivative of x^3. Substitute $u = x^3$, so $du/dx = 3x^2$ and $x^2 = \frac{1}{3}\,du/dx$. Thus

$$\int x^2 \sin(x^3)\,dx = \int \frac{1}{3}\frac{du}{dx}\sin u\,dx = \frac{1}{3}\int (\sin u)\frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C.$$

Hence $\int x^2\sin(x^3)\,dx = -\frac{1}{3}\cos(x^3) + C$.
(b) Substitute $u = 2x$, so $du/dx = 2$. Then

$$\int \sin 2x\,dx = \int \frac{1}{2}(\sin 2x)2\,dx = \frac{1}{2}\int \sin u\frac{du}{dx}\,dx$$
$$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C.$$

Thus

$$\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C. \blacktriangle$$

Example 5 Evaluate: (a) $\int \frac{x^2}{x^3+5}\,dx$, (b) $\int \frac{dt}{t^2-6t+10}$ [*Hint*: Complete the square in the denominator], and (c) $\int \sin^2 2x \cos 2x\,dx$.

Solution (a) Set $u = x^3 + 5$; $du/dx = 3x^2$. Then

$$\int \frac{x^2}{x^3+5}\,dx = \int \frac{1}{3(x^3+5)}3x^2\,dx = \frac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+5| + C.$$

(b) Completing the square (see Section R.1), we find

$$t^2 - 6t + 10 = (t^2 - 6t + 9) - 9 + 10$$
$$= (t-3)^2 + 1$$

We set $u = t - 3$; $du/dt = 1$. Then

$$\int \frac{dt}{t^2-6t+10} = \int \frac{dt}{1+(t-3)^2} = \int \frac{1}{1+u^2}\frac{du}{dt}\,dt$$
$$= \int \frac{1}{1+u^2}\,du = \tan^{-1}u + C,$$

so

$$\int \frac{dt}{t^2-6t+10} = \tan^{-1}(t-3) + C.$$

(c) We first substitute $u = 2x$, as in Example 4(b). Since $du/dx = 2$,

$$\int \sin^2 2x \cos 2x\,dx = \int \sin^2 u \cos u \frac{1}{2}\frac{du}{dx}\,dx = \frac{1}{2}\int \sin^2 u \cos u\,du.$$

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At this point, we notice that another substitution is appropriate: we set $s = \sin u$ and $ds/du = \cos u$. Then

$$\frac{1}{2}\int \sin^2 u \cos u\, du = \frac{1}{2}\int s^2 \frac{ds}{du}\, du = \frac{1}{2}\int s^2\, ds$$
$$= \frac{1}{2}\cdot\frac{1}{3}s^3 + C = \frac{s^3}{6} + C.$$

Now we must put our answer in terms of x. Since $s = \sin u$ and $u = 2x$, we have

$$\int \sin^2 2x \cos 2x\, dx = \frac{s^3}{6} + C = \frac{\sin^3 u}{6} + C = \frac{\sin^3 2x}{6} + C.$$

You should check this formula by differentiating.

You may have noticed that we could have done this problem in one step by substituting $u = \sin 2x$ in the beginning. We did the problem the long way to show that you can solve an integration problem even if you do not see everything at once. ▲

Two simple substitutions are so useful that they are worth noting explicitly. We have already used them in the preceding examples. The first is the *shifting rule*, obtained by the substitution $u = x + a$, where a is a constant. Here $du/dx = 1$.

Shifting Rule

To evaluate $\int f(x+a)\,dx$, first evaluate $\int f(u)\,du$, then substitute $x + a$ for u:

$$\int f(x+a)\,dx = F(x+a) + C, \qquad \text{where } F(u) = \int f(u)\,du.$$

The second rule is the *scaling rule*, obtained by substituting $u = bx$, where b is a constant. Here $du/dx = b$. The substitution corresponds to a change of scale on the x axis.

Scaling Rule

To evaluate $\int f(bx)\,dx$, evaluate $\int f(u)\,du$, divide by b and substitute bx for u:

$$\int f(bx)\,dx = \frac{1}{b}F(bx) + C, \qquad \text{where } F(u) = \int f(u)\,du.$$

Example 6 Find (a) $\int \sec^2(x+7)\,dx$ and (b) $\int \cos 10x\,dx$.

Solution (a) Since $\int \sec^2 u\,du = \tan u + C$, the shifting rule gives

$$\int \sec^2(x+7)\,dx = \tan(x+7) + C.$$

(b) Since $\int \cos u\,du = \sin u + C$, the scaling rule gives

$$\int \cos 10x\,dx = \tfrac{1}{10}\sin(10x) + C. \ ▲$$

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You do not need to memorize the shifting and scaling rules as such; however, the underlying substitutions are so common that you should learn to use them rapidly and accurately.

To conclude this section, we shall introduce a useful device called *differential notation*, which makes the substitution process more mechanical. In particular, this notation helps keep track of the constant factors which must be distributed between the $f(u)$ and du/dx parts of the integrand. We illustrate the device with an example before explaining why it works.

Example 7 Find $\int \frac{x^4+2}{(x^5+10x)^5}\,dx$.

Solution We wish to substitute $u = x^5 + 10x$; note that $du/dx = 5x^4 + 10$. Pretending that du/dx is a fraction, we may "solve for dx," writing $dx = du/(5x^4+10)$. Now we substitute u for $x^5 + 10x$ and $du/(5x^4+10)$ for dx in our integral to obtain

$$\int \frac{x^4+2}{(x^5+10x)^5}\,dx = \int \frac{x^4+2}{u^5}\,\frac{du}{5x^4+10} = \int \frac{x^4+2}{5(x^4+2)}\,\frac{du}{u^5} = \int \frac{1}{5}\,\frac{du}{u^5}\,.$$

Notice that the (x^4+2)'s cancelled, leaving us an integral in u which we can evaluate:

$$\frac{1}{5}\int \frac{du}{u^5} = \frac{1}{5}\left(-\frac{1}{4}u^{-4}\right) + C = -\frac{1}{20u^4} + C.$$

Substituting $x^5 + 10x$ for u gives

$$\int \frac{x^4+2}{(x^5+10x)^5}\,dx = -\frac{1}{20(x^5+10x)^4} + C. \blacktriangle$$

Although du/dx is not really a fraction, we can still justify "solving for dx" when we integrate by substitution. Suppose that we are trying to integrate $\int h(x)\,dx$ by substituting $u = g(x)$. Solving $du/dx = g'(x)$ for dx amounts to replacing dx by $du/g'(x)$ and hence writing

$$\int h(x)\,dx = \int \frac{h(x)}{g'(x)}\,du. \tag{2}$$

Now suppose that we can express $h(x)/g'(x)$ in terms of u, i.e., $h(x)/g'(x) = f(u)$ for some function f. Then we are saying that $h(x) = f(u)g'(x) = f(g(x))g'(x)$, and equation (2) just says

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$

which is the form of integration by substitution we have been using all along.

Example 8 Find $\int \left(\frac{e^{1/x}}{x^2}\right) dx$.

Solution Let $u = 1/x$; $du/dx = -1/x^2$ and $dx = -x^2\,du$, so

$$\int \left(\frac{1}{x^2}\right) e^{1/x}\,dx = \int \left(\frac{1}{x^2}\right) e^u(-x^2\,du) = -\int e^u\,du = -e^u + C$$

and therefore

$$\int \left(\frac{1}{x^2}\right) e^{1/x}\,dx = -e^{1/x} + C. \blacktriangle$$

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Integration by Substitution (Differential Notation)

To integrate $\int h(x)\,dx$ by substitution:

1. Choose a new variable $u = g(x)$.
2. Differentiate to get $du/dx = g'(x)$ and then solve for dx.
3. Replace dx in the integral by the expression found in step 2.
4. Try to express the new integrand completely in terms of u, eliminating x. (If you cannot, try another substitution or another method.)
5. Evaluate the new integral $\int f(u)\,du$ (if you can).
6. Express the result in terms of x.
7. Check by differentiating.

Example 9 (a) Calculate the following integrals: (a) $\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx$, (b) $\int \cos x[\cos(\sin x)]\,dx$, and (c) $\int \left(\frac{\sqrt{1+\ln x}}{x}\right)dx$.

Solution (a) Let $u = x^3 + 3x^2 + 1$; $du/dx = 3x^2 + 6x$, so $dx = du/(3x^2+6x)$ and

$$\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx = \int \frac{1}{\sqrt[3]{u}}\,\frac{x^2+2x}{3x^2+6x}\,du$$
$$= \frac{1}{3}\int \frac{1}{\sqrt[3]{u}}\,du = \frac{1}{3}\cdot\frac{3}{2}u^{2/3} + C.$$

Thus

$$\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx = \frac{1}{2}(x^3+3x^2+1)^{2/3} + C.$$

(b) Let $u = \sin x$; $du/dx = \cos x$, $dx = du/\cos x$, so

$$\int \cos x\big[\cos(\sin x)\big]\,dx = \int \cos x\big[\cos(\sin x)\big]\,\frac{du}{\cos x}$$
$$= \int \cos u\,du = \sin u + C,$$

and therefore

$$\int \cos x\big[\cos(\sin x)\big]\,dx = \sin(\sin x) + C.$$

(c) Let $u = 1 + \ln x$; $du/dx = 1/x$, $dx = x\,du$, so

$$\int \frac{\sqrt{1+\ln x}}{x}\,dx = \int \frac{\sqrt{1+\ln x}}{x}(x\,du) = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C,$$

and therefore

$$\int \frac{\sqrt{1+\ln x}}{x}\,dx = \frac{2}{3}(1+\ln x)^{3/2} + C. \blacktriangle$$

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Exercises for Section 7.2

Evaluate each of the integrals in Exercises 1–6 by making the indicated substitution, and check your answers by differentiating.

1. $\int 2x(x^2+4)^{3/2}\,dx$; $u = x^2+4$.
2. $\int (x+1)(x^2+2x-4)^{-4}\,dx$; $u = x^2+2x-4$.
3. $\int \dfrac{2y^7+1}{(y^8+4y-1)^2}\,dy$; $x = y^8+4y-1$.
4. $\int \dfrac{x}{1+x^4}\,dx$; $u = x^2$.
5. $\int \dfrac{\sec^2\theta}{\tan^3\theta}\,d\theta$; $u = \tan\theta$.
6. $\int \tan x\,dx$; $u = \cos x$.

Evaluate each of the integrals in Exercises 7–22 by the method of substitution, and check your answer by differentiating.

7. $\int (x+1)\cos(x^2+2x)\,dx$
8. $\int u\sin(u^2)\,du$
9. $\int \dfrac{x^3}{\sqrt{x^4+2}}\,dx$
10. $\int \dfrac{x}{(x^2+3)^2}\,dx$
11. $\int \dfrac{t^{1/3}}{(t^{4/3}+1)^{3/2}}\,dt$
12. $\int \dfrac{x^{1/2}}{(x^{3/2}+2)^2}\,dx$
13. $\int 2r\sin(r^2)\cos^3(r^2)\,dr$
14. $\int e^{\sin x}\cos x\,dx$
15. $\int \dfrac{x^3}{1+x^8}\,dx$
16. $\int \dfrac{dx}{\sqrt{1-4x^2}}$
17. $\int \sin(\theta+4)\,d\theta$
18. $\int \dfrac{1}{x^2}\sin\dfrac{1}{x}\,dx$
19. $\int (5x^4+1)(x^5+x)^{100}\,dx$
20. $\int (1+\cos s)\sqrt{s+\sin s}\,ds$
21. $\int \left(\dfrac{t+1}{\sqrt{t^2+2t+3}}\right)dt$
22. $\int \dfrac{dx}{x^2+4}$

Evaluate the indefinite integrals in Exercises 23–36.

23. $\int t\sqrt{t^2+1}\,dt$.
24. $\int t\sqrt{t+1}\,dt$.
25. $\int \cos^3\theta\,d\theta$. [*Hint*: Use $\cos^2\theta + \sin^2\theta = 1$.]
26. $\int \cot x\,dx$.
27. $\int \dfrac{dx}{x\ln x}$.
28. $\int \dfrac{dx}{\ln(x^x)}$.
29. $\int \sqrt{4-x^2}\,dx$. [*Hint*: Let $x = 2\sin u$.]
30. $\int \sin^2 x\,dx$. (Use $\cos 2x = 1 - 2\sin^2 x$.)
31. $\int \dfrac{\cos\theta}{1+\sin\theta}\,d\theta$.
32. $\int \sec^2 x(e^{\tan x}+1)\,dx$.
33. $\int \dfrac{\sin(\ln t)}{t}\,dt$.
34. $\int \dfrac{e^{2s}}{1+e^{2s}}\,ds$.
35. $\int \dfrac{\sqrt[3]{3+1/x}}{x^2}\,dx$.
36. $\int \dfrac{1}{x^3}\left(1-\dfrac{1}{x^2}\right)^{1/3}dx$.
37. Compute $\int \sin x\cos x\,dx$ by each of the following three methods: (a) Substitute $u = \sin x$, (b) substitute $u = \cos x$, (c) use the identity $\sin 2x = 2\sin x\cos x$. Show that the three answers you get are really the same.
38. Compute $\int e^{ax}\,dx$, where a is constant, by each of the following substitutions: (a) $u = ax$; (b) $u = e^x$. Show that you get the same answer either way.

★39. For which values of m and n can $\int \sin^m x\cos^n x\,dx$ be evaluated by using a substitution $u = \sin x$ or $u = \cos x$ and the identity $\cos^2 x + \sin^2 x = 1$?

★40. For which values of r can $\int \tan^r x\,dx$ be evaluated by the substitution suggested in Exercise 39?

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7.3 Changing Variables in the Definite Integral

When you change variables in a definite integral, you must keep track of the endpoints.

We have just learned how to evaluate many indefinite integrals by the method of substitution. Using the fundamental theorem of calculus, we can use this knowledge to evaluate definite integrals as well.

Example 1 Find $\int_0^2 \sqrt{x+3}\,dx$.

Solution Substitute $u = x + 3$, $du = dx$. Then

$$\int \sqrt{x+3}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x+3)^{3/2} + C.$$

By the fundamental theorem of calculus,

$$\int_0^2 \sqrt{x+3}\,dx = \frac{2}{3}(x+3)^{3/2}\Big|_0^2 = \frac{2}{3}(5^{3/2} - 3^{3/2}) \approx 3.99.$$

To check this result we observe that, on the interval $[0,2]$, $\sqrt{x+3}$ lies between $\sqrt{3}$ (≈ 1.73) and $\sqrt{5}$ (≈ 2.24), so the integral must lie between $2\sqrt{3}$ (≈ 3.46) and $2\sqrt{5}$ (≈ 4.47). (This check actually enabled the authors to spot an error in their first attempted solution of this problem.) ▲

Notice that we must express the indefinite integral in terms of x before plugging in the endpoints 0 and 2, since they refer to values of x. It is possible, however, to evaluate the definite integral directly in the u variable—*provided that we change the endpoints*. We offer an example before stating the general procedure.

Example 2 Find $\int_1^4 \frac{x}{1+x^4}\,dx$.

Solution Substitute $u = x^2$, $du = 2x\,dx$, that is, $x\,dx = du/2$. As x runs from 1 to 4, $u = x^2$ runs from 1 to 16, so we have

$$\int_1^4 \frac{x}{1+x^4}\,dx = \int_1^{16} \frac{x}{1+x^4}\frac{du}{2x} = \frac{1}{2}\int_1^{16}\frac{du}{1+u^2}$$

$$= \frac{1}{2}\tan^{-1}u\Big|_1^{16} = \frac{1}{2}(\tan^{-1}16 - \tan^{-1}1) \approx 0.361. \quad ▲$$

In general, suppose that we have an integral of the form $\int_a^b f(g(x))g'(x)\,dx$. If $F'(u) = f(u)$, then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$; by the fundamental theorem of calculus, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

However, the right-hand side is equal to $\int_{g(a)}^{g(b)} f(u)\,du$, so we have the formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Notice that $g(a)$ and $g(b)$ are the values of $u = g(x)$ when $x = a$ and b, respectively. Thus we can evaluate an integral $\int_a^b h(x)\,dx$ by writing $h(x)$ as

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$f(g(x))g'(x)$ and using the formula

$$\int_a^b h(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 3 Evaluate $\int_0^{\pi/4} \cos 2\theta\, d\theta$.

Solution Let $u = 2\theta$; $d\theta = \frac{1}{2}\,du$; $u = 0$ when $\theta = 0$, $u = \pi/2$ when $\theta = \pi/4$. Thus

$$\int_0^{\pi/4} \cos 2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \cos u\, du = \frac{1}{2}\sin u\Big|_0^{\pi/2} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}\,. \blacktriangle$$

Definite Integral by Substitution

Given an integral $\int_a^b h(x)\,dx$ and a new variable $u = g(x)$:

1. Substitute $du/g'(x)$ for dx and then try to express the integrand $h(x)/g'(x)$ in terms of u.
2. Change the endpoints a and b to $g(a)$ and $g(b)$, the corresponding values of u.

Then

$$\int_a^b h(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

where $f(u) = h(x)/(du/dx)$. Since $h(x) = f(g(x))g'(x)$, this can be written as

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 4 Evaluate $\int_1^5 \frac{x}{x^4 + 10x^2 + 25}\,dx$.

Solution Seeing that the denominator can be written in terms of x^2, we try $u = x^2$, $dx = du/(2x)$; $u = 1$ when $x = 1$ and $u = 25$ when $x = 5$. Thus

$$\int_1^5 \frac{x}{x^4 + 10x^2 + 25}\,dx = \frac{1}{2}\int_1^{25} \frac{du}{u^2 + 10u + 25}\,.$$

Now we notice that the denominator is $(u + 5)^2$, so we set $v = u + 5$, $du = dv$; $v = 6$ when $u = 1$, $v = 30$ when $u = 25$. Therefore

$$\frac{1}{2}\int_1^{25} \frac{du}{u^2 + 10u + 25} = \frac{1}{2}\int_6^{30} \frac{dv}{v^2} = \frac{1}{2}\left(-\frac{1}{v}\right)\Big|_6^{30}$$
$$= -\frac{1}{60} + \frac{1}{12} = \frac{1}{15}\,.$$

If you see the substitution $v = x^2 + 5$ right away, you can do the problem in one step instead of two. $\blacktriangle$

Example 5 Find $\int_0^{\pi/4} (\cos^2\theta - \sin^2\theta)\,d\theta$.

Solution It is not obvious what substitution is appropriate here, so a little trial and error is called for. If we remember the trigonometric identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

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we can proceed easily:

$$\int_0^{\pi/4}(\cos^2\theta - \sin^2\theta)\,d\theta = \int_0^{\pi/4}\cos 2\theta\,d\theta = \int_0^{\pi/2}\cos u\,\frac{du}{2} \qquad (u = 2\theta)$$

$$= \frac{\sin u}{2}\bigg|_0^{\pi/2} = \frac{1-0}{2} = \frac{1}{2}.$$

(See Exercise 32 for another method.) ▲

Example 6 Evaluate $\int_0^1 \frac{e^x}{1+e^x}\,dx$.

Solution Let $u = 1 + e^x$; $du = e^x\,dx$, $dx = du/e^x$; $u = 1 + e^0 = 2$ when $x = 0$ and $u = 1 + e$ when $x = 1$. Thus

$$\int_0^1 \frac{e^x}{1+e^x}\,dx = \int_2^{1+e}\frac{1}{u}\,du = \ln u\bigg|_2^{1+e} = \ln(1+e) - \ln 2 = \ln\left(\frac{1+e}{2}\right). \ ▲$$

Substitution does not always work. We can always make a substitution, but sometimes it leads nowhere.

Example 7 What does the integral $\int_0^2 \frac{dx}{1+x^4}$ become if you substitute $u = x^2$?

Solution If $u = x^2$, $du/dx = 2x$ and $dx = du/2x$, so

$$\int_0^2 \frac{dx}{1+x^4} = \int_0^4 \frac{1}{1+u^2}\,\frac{du}{2x}.$$

We must solve $u = x^2$ for x; since $x \geqslant 0$, we get $x = \sqrt{u}$, so

$$\int_0^2 \frac{dx}{1+x^4} = \int_0^4 \frac{du}{2\sqrt{u}\,(1+u^2)}.$$

Unfortunately, we do not know how to evaluate the integral in u, so all we have done is to equate two unknown quantities. ▲

As in Example 7, after a substitution, the integral $\int f(u)\,du$ might still be something we do not know how to evaluate. In that case it may be necessary to make another substitution or use a completely different method. There is an infinite choice of substitutions available in any given situation. It takes practice to learn to choose one that works.

In general, integration is a trial-and-error process that involves a certain amount of educated guessing. What is more, the antiderivatives of such innocent-looking functions as

$$\frac{1}{\sqrt{(1-x^2)(1-2x^2)}} \quad \text{and} \quad \frac{1}{\sqrt{3-\sin^2 x}}$$

cannot be expressed in any way as algebraic combinations and compositions of polynomials, trigonometric functions, or exponential functions. (The proof of a statement like this is not elementary; it belongs to a subject known as "differential algebra".) Despite these difficulties, you can learn to integrate many functions, but the learning process is slower than for differentiation, and practice is more important than ever.

Since integration is harder than differentiation, one often uses tables of integrals. A short table is available on the endpapers of this book, and extensive books of tables are on the market. (Two of the most popular are Burington's and the CRC tables, both of which contain a great deal of mathematical data in addition to the integrals.) Using these tables requires a

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knowledge of the basic integration techniques, though, and that is why you still need to learn them.

Example 8 Evaluate $\int_1^3 \frac{dx}{x\sqrt{1+x}}$ using the tables of integrals.

Solution We search the tables for a form similar to this and find number 49 with $a = 1$, $b = 1$. Thus

$$\int \frac{dx}{x\sqrt{1+x}} = \ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + C.$$

Hence

$$\int_1^3 \frac{dx}{x\sqrt{1+x}} = \ln\left|\frac{\sqrt{4}-1}{\sqrt{4}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| = \ln\frac{1}{3} - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|$$

$$= \ln\left[\frac{\sqrt{2}+1}{3(\sqrt{2}-1)}\right] = \ln\left(1 + \frac{2}{3}\sqrt{2}\right). \blacktriangle$$

Exercises for Section 7.3

Evaluate the definite integrals in Exercises 1–22.

1. $\int_{-1}^1 \sqrt{x+2}\,dx$
2. $\int_2^3 \frac{dt}{t-1}$
3. $\int_0^2 x\sqrt{x^2+1}\,dx$
4. $\int_0^1 t\sqrt{t^2+1}\,dt$
5. $\int_2^4 (x+1)(x^2+2x+1)^{5/4}\,dx$
6. $\int_1^2 \frac{\sqrt{1+\ln x}}{x}\,dx$
7. $\int_1^3 \frac{3x}{(x^2+5)^2}\,dx$
8. $\int_1^2 \frac{t^2+1}{\sqrt{t^3+3t+3}}\,dt$
9. $\int_0^1 xe^{(x^2)}\,dx$
10. $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$
11. $\int_0^{\pi/6} \sin(3\theta+\pi)\,d\theta$
12. $\int_0^{\pi} \sin(\theta/2+\pi/4)\,d\theta$
13. $\int_{-\pi/2}^{\pi/2} 5\cos^2 x \sin x\,dx.$
14. $\int_{\pi/4}^{\pi/2} \frac{\csc^2 y}{\cot^2 y + 2\cot y + 1}\,dy.$
15. $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx.$
16. $\int_0^1 \frac{x^2}{x^3+1}\,dx.$
17. $\int_{\pi/8}^{\pi/4} \tan\theta\,d\theta.$
18. $\int_{\pi/4}^{\pi/2} \cot\theta\,d\theta.$
19. $\int_0^{\pi/2} \sin x\cos x\,dx.$
20. $\int_1^{\pi/2} [\ln(\sin x) + (x\cot x)](\sin x)^x\,dx.$
21. $\int_1^3 \frac{x^3+x-1}{x^2+1}\,dx$ (simplify first).
22. $\int_1^e \frac{2\ln(x^x)+1}{x^2}\,dx.$
23. Using the result $\int_0^{\pi/2} \sin^2 x\,dx = \pi/4$ (See Exercise 57, Section 7.1), compute each of the following integrals: (a) $\int_0^{\pi} \sin^2(x/2)\,dx$; (b) $\int_{\pi/2}^{\pi} \sin^2(x-\pi/2)\,dx$; (c) $\int_0^{\pi/4} \cos^2(2x)\,dx.$
24. (a) By combining the shifting and scaling rules, find a formula for $\int f(ax+b)\,dx$.
    (b) Find $\int_2^3 \frac{dx}{4x^2+12x+9}$ [*Hint*: Factor the denominator.]
25. What happens in the integral
$$\int_0^1 \frac{(x^2+3x)}{\sqrt[3]{x^3+3x^2+1}}\,dx$$
if you make the substitution $u = x^3+3x^2+1$?
26. What becomes of the integral $\int_0^{\pi/2} \cos^4 x\,dx$ if you make the substitution $u = \cos x$?

Evaluate the integrals in Exercises 27–30 using the tables.

27. $\int_0^1 \frac{dx}{3x^2+2x+1}$
28. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$
29. $\int_0^1 \frac{dx}{\sqrt{3x^2+2x+1}}$
30. $\int_2^3 \frac{\sqrt{x^2-2}}{x^4}\,dx$

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31. Given two functions f and g, define a function h by
$$h(x) = \int_0^1 f(x-t)g(t)\,dt.$$
Show that
$$h(x) = \int_{x-1}^{x} g(x-t)f(t)\,dt.$$
32. Give another solution to Example 5 by writing $\cos^2\theta - \sin^2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$ and using the substitution $u = \cos\theta + \sin\theta$.
33. Find the area under the graph of the function $y = (x+1)/(x^2+2x+2)^{3/2}$ from $x=0$ to $x=1$.
34. The curve $x^2/a^2 + y^2/b^2 = 1$, where a and b are positive, describes an *ellipse* (Fig. 7.3.1). Find the area of the region inside this ellipse. [*Hint*: Write half the area as an integral and then change variables in the integral so that it becomes the integral for the area inside a semicircle.]

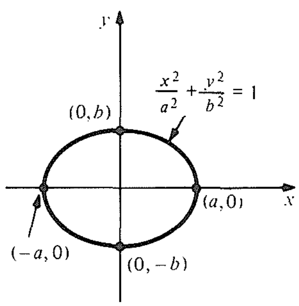

Figure 7.3.1. Find the area inside the ellipse.

35. The curve $y = x^{1/3}$, $1 \leqslant x \leqslant 8$, is revolved about the y axis to generate a surface of revolution of area s. In Chapter 10 we will prove that the area is given by $s = \int_1^2 2\pi y^3\sqrt{1+9y^4}\,dy$. Evaluate this integral.

★36. Let $f(x) = \int_1^x (dt/t)$. Show, using substitution, and without using logarithms, that $f(a)+f(b) = f(ab)$ if $a, b > 0$. [*Hint*: Transform $\int_a^{ab}\frac{dt}{t}$ by a change of variables.]

37. (a) Find $\int_0^{\pi/2}\cos^2 x\sin x\,dx$ by substituting $u = \cos x$ and changing the endpoints.

★(b) Is the formula
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$
valid if $a < b$, yet $g(a) > g(b)$? Discuss.

7.4 Integration by Parts

Integrating the product rule leads to the method of integration by parts.

The second of the two important new methods of integration is developed in this section. The method parallels that of substitution, with the chain rule replaced by the product rule.

The product rule for derivatives asserts that
$$(FG)'(x) = F'(x)G(x) + F(x)G'(x).$$
Since $F(x)G(x)$ is an antiderivative for $F'(x)G(x) + F(x)G'(x)$, we can write
$$\int [F'(x)G(x) + F(x)G'(x)]\,dx = F(x)G(x) + C.$$
Applying the sum rule and transposing one term leads to the formula
$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx + C.$$
If the integral on the right-hand side can be evaluated, it will have its own constant C, so it need not be repeated. We thus write
$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx, \tag{1}$$
which is the formula for integration by parts. To apply formula (1) we need to break up a given integrand as a product $F(x)G'(x)$, write down the right-hand side of formula (1), and hope that we can integrate $F'(x)G(x)$. Integrands involving trigonometric, logarithmic, and exponential functions are often good candidates for integration by parts, but practice is necessary to learn the best way to break up an integrand as a product.

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Example 1 Evaluate $\int x \cos x \, dx$.

Solution If we remember that $\cos x$ is the derivative of $\sin x$, we can write $x \cos x$ as $F(x)G'(x)$, where $F(x) = x$ and $G(x) = \sin x$. Applying formula (1), we have

$$\int x \cos x \, dx = x \cdot \sin x - \int 1 \cdot \sin x \, dx = x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + C.$$

Checking by differentiation, we have

$$\frac{d}{dx}(x \sin x + \cos x) = x \cos x + \sin x - \sin x = x \cos x,$$

as required. ▲

It is often convenient to write formula (1) using differential notation. Here we write $u = F(x)$ and $v = G(x)$. Then $du/dx = F'(x)$ and $dv/dx = G'(x)$. Treating the derivatives as if they were quotients of "differentials" du, dv, and dx, we have $du = F'(x)\,dx$ and $dv = G'(x)\,dx$. Substituting these into formula (1) gives

$$\int u \, dv = uv - \int v \, du \tag{2}$$

(see Fig. 7.4.1).

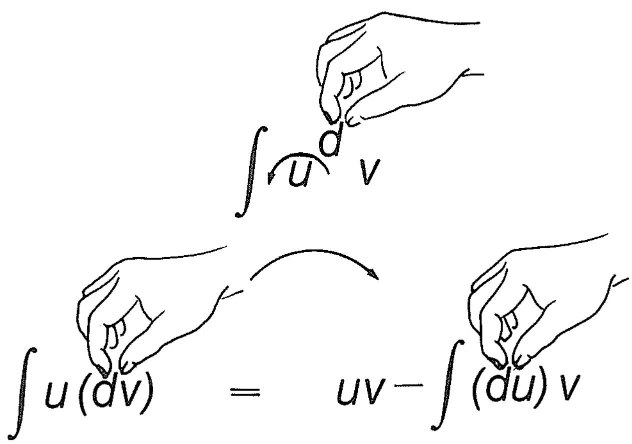

Figure 7.4.1. You may move "d" from v to u if you switch the sign and add uv.

Integration by Parts

To evaluate $\int h(x)\,dx$ by parts:

1. Write $h(x)$ as a product $F(x)G'(x)$, where the antiderivative $G(x)$ of $G'(x)$ is known.
2. Take the derivative $F'(x)$ of $F(x)$.
3. Use the formula

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx,$$

i.e., with $u = F(x)$ and $v = G(x)$,

$$\int u \, dv = uv - \int v \, du.$$

When you use integration by parts, to integrate a function h write $h(x)$ as a product $F(x)G'(x) = u\,dv/dx$; the factor $G'(x)$ is a function whose antideriv-

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ative $v = G(x)$ can be found. With a good choice of $u = F(x)$ and $v = G(x)$, the integral $\int F'(x)G(x)\,dx = \int v\,du$ becomes simpler than the original problem $\int u\,dv$. The ability to make good choices of u and v comes with practice. A last reminder—don't forget the minus sign.

Example 2 Find (a) $\int x\sin x\,dx$ and (b) $\int x^2\sin x\,dx$.

Solution (a) (*Using formula* (1)) Let $F(x) = x$ and $G'(x) = \sin x$. Integrating $G'(x)$ gives $G(x) = -\cos x$; also, $F'(x) = 1$, so

$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x - (-\sin x) + C \\ &= -x\cos x + \sin x + C.\end{aligned}$$

(b) (*Using formula* (2)) Let $u = x^2$, $dv = \sin x\,dx$. To apply formula (2) for integration by parts, we need to know v. But $v = \int dv = \int \sin x\,dx = -\cos x$. (We leave out the arbitrary constant here and will put it in at the end of the problem.)

Now

$$\begin{aligned}\int x^2\sin x\,dx &= uv - \int v\,du \\ &= -x^2\cos x - \int -\cos x\cdot 2x\,dx \\ &= -x^2\cos x + 2\int x\cos x\,dx.\end{aligned}$$

Using the result of Example 1, we obtain

$$-x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Check this result by differentiating—it is nice to see all the cancellation. ▲

Integration by parts is also commonly used in integrals involving e^x and $\ln x$.

Example 3 (a) Find $\int \ln x\,dx$ using integration by parts. (b) Find $\int xe^x\,dx$.

Solution (a) Here, let $u = \ln x$, $dv = 1\,dx$. Then $du = dx/x$ and $v = \int 1\,dx = x$. Applying the formula for integration by parts, we have

$$\begin{aligned}\int \ln x\,dx &= uv - \int v\,du = (\ln x)x - \int x\frac{dx}{x} \\ &= x\ln x - \int 1\,dx = x\ln x - x + C.\end{aligned}$$

(Compare Example 7, Section 7.1.)

(b) Let $u = x$ and $v = e^x$, so $dv = e^x\,dx$. Thus, using integration by parts,

$$\begin{aligned}\int xe^x\,dx &= \int u\,dv = uv - \int v\,du \\ &= xe^x - \int e^x\,dx = xe^x - e^x + C.\ ▲\end{aligned}$$

Next we consider an example involving both e^x and $\sin x$.

Example 4 Apply integration by parts twice to find $\int e^x\sin x\,dx$.

Solution Let $u = \sin x$ and $v = e^x$, so $dv = e^x\,dx$ and

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx. \tag{3}$$

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Repeating the integration by parts,

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx, \tag{4}$$

where, this time, $u = \cos x$ and $v = e^x$. Substituting formula (4) into (3), we get

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx.$$

The unknown integral $\int e^x \sin x\,dx$ appears twice in this equation. Writing "I" for this integral, we have

$$I = e^x \sin x - e^x \cos x - I,$$

and solving for I gives

$$I = \tfrac{1}{2} e^x (\sin x - \cos x),$$

i.e.,

$$\int e^x \sin x\,dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C.$$

Some students like to remember this as "the I method." ▲

Some special purely algebraic expressions can also be handled by a clever use of integration by parts, as in the next example.

Example 5 Find $\int x^7 (x^4 + 1)^{2/3}\,dx$.

Solution By taking x^3 out of x^7 and grouping it with $(x^4 + 1)^{2/3}$, we get an expression which we can integrate. Specifically, we set $dv = 4x^3(x^4 + 1)^{2/3}\,dx$, leaving $u = x^4/4$. Using integration by substitution, we get $v = \frac{3}{5}(x^4 + 1)^{5/3}$, and differentiating, we get $du = x^3\,dx$. Hence

$$\int x^7 (x^4 + 1)^{2/3}\,dx = \frac{3x^4}{20}(x^4 + 1)^{5/3} - \frac{3}{5}\int x^3 (x^4 + 1)^{5/3}\,dx.$$

Substituting $w = (x^4 + 1)$ gives

$$\int x^3 (x^4 + 1)^{5/3}\,dx = \tfrac{3}{32}(x^4 + 1)^{8/3} + C;$$

hence

$$\begin{aligned}\int x^7 (x^4 + 1)^{2/3}\,dx &= \tfrac{3}{20} x^4 (x^4 + 1)^{5/3} - \tfrac{9}{160}(x^4 + 1)^{8/3} + C \\ &= \tfrac{3}{160}(x^4 + 1)^{5/3}(5x^4 - 3) + C.\ ▲\end{aligned}$$

Using integration by parts and then the fundamental theorem of calculus, we can calculate definite integrals.

Example 6 Find $\int_{-\pi/2}^{\pi/2} x \sin x\,dx$.

Solution From Example 2 (a) we have $\int x \sin x\,dx = -x \cos x + \sin x + C$, so

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} x \sin x\,dx &= (-x \cos x + \sin x)\Big|_{-\pi/2}^{\pi/2} \\ &= \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left[\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)\right] \\ &= (0 + 1) - [0 + (-1)] = 2.\ ▲\end{aligned}$$

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Example 7 Find (a) $\int_0^{\ln 2} e^x \ln(e^x+1)\,dx$ and (b) $\int_1^e \sin(\ln x)\,dx$.

Solution (a) Notice that e^x is the derivative of (e^x+1), so we first make the substitution $t = e^x + 1$. Then

$$\int_0^{\ln 2} e^x \ln(e^x+1)\,dx = \int_2^3 \ln t\,dt,$$

and, from Example 3, $\int \ln t\,dt = t\ln t - t + C$. Therefore

$$\int_0^{\ln 2} e^x \ln(e^x+1)\,dx = (t\ln t - t)\Big|_2^3 = (3\ln 3 - 3) - (2\ln 2 - 2)$$
$$= 3\ln 3 - 2\ln 2 - 1 \approx 0.9095.$$

(b) Again we begin with a substitution. Let $u = \ln x$, so that $x = e^u$ and $du = (1/x)\,dx$. Then $\int \sin(\ln x)\,dx = \int (\sin u)e^u\,du$, which was evaluated in Example 4. Hence

$$\int_1^e \sin(\ln x)\,dx = \int_0^1 e^u \sin u\,du = \frac{1}{2}e^u(\sin u - \cos u)\Big|_0^1$$
$$= \left[\frac{1}{2}e^1(\sin 1 - \cos 1)\right] - \left[\frac{1}{2}e^0(\sin 0 - \cos 0)\right]$$
$$= \frac{e}{2}\left(\sin 1 - \cos 1 + \frac{1}{e}\right). \blacktriangle$$

Example 8 Find the area under the nth bend of $y = x\sin x$ in the first quadrant (see Fig. 7.4.2).

Solution The nth bend occurs between $x = (2n-2)\pi$ and $(2n-1)\pi$. (Check $n = 1$ and $n = 2$ with the figure.) The area under this bend can be evaluated using integration by parts [Example 2(a)]:

$$\int_{(2n-2)\pi}^{(2n-1)\pi} x\sin x\,dx = -x\cos x + \sin x\Big|_{(2n-2)\pi}^{(2n-1)\pi}$$
$$= -(2n-1)\pi\cos[(2n-1)\pi] + \sin[(2n-1)\pi]$$
$$+ (2n-2)\pi\cos(2n-2)\pi - \sin(2n-2)\pi$$
$$= -(2n-1)\pi(-1) + 0 + (2n-2)\pi(1) - 0$$
$$= (2n-1)\pi + (2n-2)\pi = (4n-3)\pi.$$

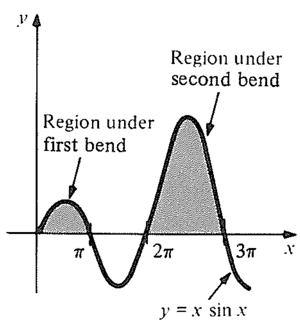

Figure 7.4.2. What is the area under the nth bend?

Thus the areas under successive bends are π, 5π, 9π, 13π, and so forth. ▲

We shall now use integration by parts to obtain a formula for the integral of the inverse of a function.

If f is a differentiable function, we write $f(x) = 1\cdot f(x)$; then

$$\int f(x)\,dx = \int 1\cdot f(x)\,dx = xf(x) - \int xf'(x)\,dx. \qquad (5)$$

Introducing $y = f(x)$ as a new variable, with $dx = dy/f'(x)$, we get

$$\int f(x)\,dx = xy - \int x\,dy. \qquad (6)$$

Assuming that f has an inverse function g, we have $x = g(y)$, and equation (6) becomes

$$\int f(x)\,dx = xf(x) - \int g(y)\,dy. \qquad (7)$$

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Thus we can integrate f if we know how to integrate its inverse. In the notation $y = f(x)$, equation (7) becomes

$$\int y\,dx = xy - \int x\,dy. \tag{8}$$

Notice that equation (8) looks just like the formula for integration by parts, but we are now considering x and y as functions of one another rather than as two functions of a third variable.

Example 9 Use equation (8) to compute $\int \ln x\,dx$.

Solution Viewing $y = \ln x$ as the inverse function of $x = e^y$, equation (8) reads

$$\int \ln x\,dx = xy - \int e^y\,dy = x\ln x - e^y + C = x\ln x - x + C,$$

which is the same result (and essentially the same method) as in Example 3. ▲

We can also state our result in terms of antiderivatives. If $G(y)$ is an antiderivative for $g(y)$, then

$$F(x) = xf(x) - G(f(x)) \tag{9}$$

is an antiderivative for f. (This can be checked by differentiation.)

Example 10 (a) Find an antiderivative for $\cos^{-1}x$. (b) Find $\int \csc^{-1}\sqrt{x}\,dx$.

Solution (a) If $f(x) = \cos^{-1}x$, then $g(y) = \cos y$ and $G(y) = \sin y$. By formula (9),

$$F(x) = x\cos^{-1}x - \sin(\cos^{-1}x);$$

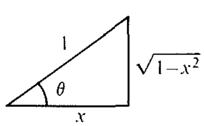

Figure 7.4.3. $\sin(\cos^{-1}x) = \sqrt{1-x^2}$

But $\sin(\cos^{-1}x) = \sqrt{1-x^2}$ (Fig. 7.4.3), so

$$F(x) = x\cos^{-1}x - \sqrt{1-x^2}$$

is an antiderivative for $\cos^{-1}x$. This may be checked by differentiation.

(b) If $y = \csc^{-1}\sqrt{x}$, we have $\csc y = \sqrt{x}$ and $x = \csc^2 y$. Then

$$\begin{aligned}
\int \csc^{-1}\sqrt{x}\,dx &= \int y\,dx = xy - \int x\,dy \\
&= x\csc^{-1}\sqrt{x} - \int \csc^2 y\,dy \\
&= x\csc^{-1}\sqrt{x} + \cot y + C \\
&= x\csc^{-1}\sqrt{x} + \cot\left(\csc^{-1}\sqrt{x}\right) + C \\
&= x\csc^{-1}\sqrt{x} + \sqrt{x-1} + C \qquad \text{(see Fig. 7.4.4).}
\end{aligned}$$
▲

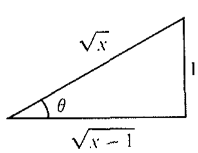

Figure 7.4.4. $\theta = \csc^{-1}\sqrt{x}$.

Example 11 (a) Find $\int \sqrt{\sqrt{x}+1}\,dx$. (b) Find $\int x\cos^{-1}x\,dx$, $0 < x < 1$.

Solution (a) If $y = \sqrt{\sqrt{x}+1}$, then $y^2 = \sqrt{x}+1$, $\sqrt{x} = y^2 - 1$, and $x = (y^2-1)^2$. Thus we have

$$\begin{aligned}
\int \sqrt{\sqrt{x}+1}\,dx &= xy - \int x\,dy = x\sqrt{\sqrt{x}+1} - \int (y^4 - 2y^2 + 1)\,dy \\
&= x\sqrt{\sqrt{x}+1} - \frac{1}{5}y^5 + \frac{2}{3}y^3 - y + C \\
&= x\sqrt{\sqrt{x}+1} - \frac{1}{5}\left(\sqrt{x}+1\right)^{5/2} + \frac{2}{3}\left(\sqrt{x}+1\right)^{3/2} \\
&\quad - \left(\sqrt{x}+1\right)^{1/2} + C.
\end{aligned}$$

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(b) Integrating by parts,

$$\int x\cos^{-1}x\,dx = \frac{x^2}{2}\cos^{-1}x + \int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx.$$

The last integral may be evaluated by letting $x = \cos u$:

$$\int\frac{x^2}{\sqrt{1-x^2}}\,dx = -\int\frac{\cos^2 u}{\sin u}\sin u\,du = -\int\cos^2 u\,du$$

But $\cos^2 u = \dfrac{\cos 2u + 1}{2}$, so

$$\int\cos^2 u\,du = \frac{1}{4}\sin 2u + \frac{u}{2} + C = \frac{1}{2}\sin u\cos u + \frac{u}{2} + C.$$

Thus,

$$\int x\cos^{-1}x\,dx = \frac{x^2}{2}\cos^{-1}x - \frac{1}{4}\sin(\cos^{-1}x)x - \frac{\cos^{-1}x}{4} + C$$
$$= \frac{x^2}{2}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} - \frac{1}{4}\cos^{-1}x + C. \blacktriangle$$

Exercises for Section 7.4

Evaluate the indefinite integrals in Exercises 1–26 using integration by parts.

1. $\int(x+1)\cos x\,dx$
2. $\int(x-2)\sin x\,dx$
3. $\int x\cos(5x)\,dx$
4. $\int x\sin(10x)\,dx$
5. $\int x^2\cos x\,dx$
6. $\int x^2\sin x\,dx$
7. $\int(x+2)e^x\,dx$
8. $\int(x^2-1)e^{2x}\,dx$
9. $\int\ln(10x)\,dx$
10. $\int x\ln x\,dx$
11. $\int x^2\ln x\,dx$
12. $\int\ln(9+x^2)\,dx$
13. $\int s^2e^{3s}\,ds$
14. $\int(s+1)^2e^s\,ds.$
15. $\int\dfrac{x^5}{(x^3-4)^{2/3}}\,dx$
16. $\int\dfrac{x^2}{(x^2+1)^2}\,dx$
17. $\int 2t^3\cos t^2\,dt$
18. $\int\dfrac{x^3}{\sqrt{x^2+1}}\,dx$
19. $\int\dfrac{1}{x^3}\cos\dfrac{1}{x}\,dx$
20. $\int x\sin(\ln x)\,dx$
21. $\int\tan x\ln(\cos x)\,dx$
22. $\int e^{2x}e^{(e^x)}\,dx$
23. $\int\cos^{-1}(2x)\,dx$
24. $\int\sin^{-1}x\,dx$
25. $\int\sqrt{\dfrac{1}{y}-1}\;dy$
26. $\int(\sqrt{x}-2)^{1/5}\,dx$
27. Find $\int\sin x\cos x\,dx$ by using integration by parts with $u=\sin x$ and $dv=\cos x\,dx$. Compare the result with substituting $u=\sin x$.
28. Compute $\int\sqrt{x}\,dx$ by the rule for inverse functions. Compare with the result given by the power rule.
29. What happens in Example 2(a) if you choose $F'(x)=x$ and $G(x)=\sin x$?
30. What would have happened in Example 5, if in the integral $\int e^x\cos x\,dx$ obtained in the first integral by parts, you had taken $u=e^x$ and $v=\sin x$ and integrated by parts a second time?

Evaluate the definite integrals in Exercises 31–46.

31. $\int_0^{\pi/5}(8+5\theta)(\sin 5\theta)\,d\theta$
32. $\int_1^2 x\ln x\,dx$
33. $\int_1^3\ln x^3\,dx$
34. $\int_0^1 xe^x\,dx$
35. $\int_0^{\pi/4}(x^2+x-1)\cos x\,dx$
36. $\int_0^{\pi/2}\sin 3x\cos 2x\,dx$
37. $\int_{1/8}^{1/4}\cos^{-1}(4x)\,dx$
38. $\int_0^1 x\tan^{-1}x\,dx$
39. $\int_1^e(\ln x)^2\,dx$
40. $\int_0^{\pi/2}\sin 2x\cos x\,dx.$
41. $\int_{-\pi}^{\pi}e^{2x}\sin(2x)\,dx.$
42. $\int_0^{\pi^2}\sin\sqrt{x}\,dx.$ [*Hint*: Change variables first.]
43. $\int_1^2 x^{1/3}(x^{2/3}+1)^{3/2}\,dx.$
44. $\int_0^1\dfrac{x^3}{(x^2+1)^{1/2}}\,dx.$
45. $\int_0^{1/2\sqrt{2}}\sin^{-1}2x\,dx.$
46. $\int_0^1\cos^{-1}(\sqrt{y})\,dy.$

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47. Show that
$$\int_0^1 \sqrt{2-x^2}\,dx - \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = (1-\pi/2)/2.$$

48. Find $\int_2^{34} f(x)\,dx$, where f is the inverse function of $g(y) = y^5 + y$.

49. Find $\int_0^{2\pi} x \sin ax\,dx$ as a function of a. What happens to this integral as a becomes larger and larger? Can you explain why?

50. (a) Integrating by parts twice (see Example 4), find $\int \sin ax \cos bx\,dx$, where $a^2 \neq b^2$.
(b) Using the formula $\sin 2x = 2\sin x \cos x$, find $\int \sin ax \cos bx\,dx$ when $a = \pm b$.
(c) Let $g(a) = (4/\pi)\int_0^{\pi/2} \sin x \sin ax\,dx$. Find a formula for $g(a)$. (The formula will have to distinguish the cases $a^2 \neq 1$ and $a^2 = 1$.)
(d) Evaluate $g(a)$ for $a = 0.9$, 0.99, 0.999, 0.9999, and so on. Compare the results with $g(1)$. Also try $a = 1.1, 1.01, 1.001$, and so on. What do you guess is true about the function g at $a = 1$?

51. (a) Integrating by parts twice, show that
$$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b\sin bx + a\cos bx}{a^2+b^2}\right) + C.$$
(b) Evaluate $\int_0^{\pi/10} e^{3x}\cos 5x\,dx$.

52. (a) Prove the following *reduction formula*:
$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx.$$
(b) Evaluate $\int_0^3 x^3 e^x\,dx$

53. (a) Prove the following *reduction formula*:
$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$
(b) Use part (a) to show that
$$\int \cos^2 x\,dx = \frac{1}{2}(\cos x \sin x + x) + C$$
and
$$\int \cos^4 x\,dx = \frac{1}{4}\left(\cos^3 x \sin x + \frac{3}{2}\cos x \sin x + \frac{3x}{2}\right) + C.$$

54. The mass density of a beam is $\rho = x^2 e^{-x}$ kilograms per centimeter. The beam is 200 centimeters long, so its mass is $M = \int_0^{200} \rho\,dx$ kilograms. Find the value of M.

55. The volume of the solid formed by rotation of the plane region enclosed by $y = 0$, $y = \sin x$, $x = 0$, $x = \pi$, around the y axis, will be shown in Chapter 9 to be given by $V = \int_0^{\pi} 2\pi x \sin x\,dx$. Find V.

56. The *Fourier series* analysis of the sawtooth wave requires the computation of the integral
$$b_m = \frac{\omega^2 A}{2\pi^2}\int_{-\pi/\omega}^{\pi/\omega} t\sin(m\omega t)\,dt,$$
where m is an integer and ω and A are nonzero constants. Compute it.

57. The current i in an underdamped RLC circuit is given by
$$i = EC\left(\frac{\alpha^2}{\omega} + \omega\right)e^{-\alpha t}\sin(\omega t).$$
The constants are E = constant emf, switched on at $t = 0$, C = capacitance in farads, R = resistance in ohms, L = inductance in henrys, $\alpha = R/2L$, $\omega = (1/2L)(4L/C - R^2)^{1/2}$.
(a) The charge Q in coulombs is given by $dQ/dt = i$, and $Q(0) = 0$. Find an integral formula for Q, using the fundamental theorem of calculus.
(b) Determine Q by integration.

58. A critically damped RLC circuit with a steady emf of E volts has current $i = EC\alpha^2 te^{-\alpha t}$, where $\alpha = R/2L$. The constants R, L, C are in ohms, henrys, and farads, respectively. The charge Q in coulombs is given by $Q(T) = \int_0^T i\,dt$. Find it explicitly, using integration by parts.

★59. Draw a figure to illustrate the formula for integration of inverse functions:
$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy,$$
where $0 < a < b$, $0 < f(a) < f(b)$, f is increasing on $[a,b]$, and g is the inverse function of f.

★60. (a) Suppose that $\phi'(x) > 0$ for all x in $[0,\infty)$ and $\phi(0) = 0$. Show that if $a \geqslant 0$, $b \geqslant 0$, and b is in the domain of ϕ^{-1}, then *Young's inequality* holds:
$$ab \leqslant \int_0^a \phi(x)\,dx + \int_0^b \phi^{-1}(y)\,dy,$$
where ϕ^{-1} is the inverse function to ϕ. [*Hint*: Express $\int_0^b \phi^{-1}(y)\,dy$ in terms of an integral of ϕ by using the formula for integrating an inverse function. Consider separately the cases $\phi(a) \leqslant b$ and $\phi(a) \geqslant b$. For the latter, prove the inequality $\int_{\phi^{-1}(b)}^a \phi(x)\,dx \geqslant \int_{\phi^{-1}(b)}^a b\,dx = b[a - \phi^{-1}(b)]$.]
(b) Prove (a) by a geometric argument based on Exercise 59.
(c) Using the result of part (a), show that if $a, b \geqslant 0$ and $p, q > 1$, with $1/p + 1/q = 1$, then *Minkowski's inequality* holds:
$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

★61. If f is a function on $[0, 2\pi]$, the numbers
$$a_n = (1/\pi)\int_0^{2\pi} f(x)\cos nx\,dx,$$
$$b_n = (1/\pi)\int_0^{2\pi} f(x)\sin nx\,dx$$
are called the *Fourier coefficients* of f ($n = 0, \pm 1, \pm 2, \ldots$). Find the Fourier coefficients of: (a) $f(x) = 1$; (b) $f(x) = x$; (c) $f(x) = x^2$; (d) $f(x) = \sin 2x + \sin 3x + \cos 4x$.

★62. Following Example 5, find a general formula for $\int x^{2n-1}(x^n + 1)^m\,dx$, where n and m are rational numbers with $n \neq 0$, $m \neq -1, -2$.

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Review Exercises for Chapter 7

Evaluate the integrals in Exercises 1–46.

1. $\int (x + \sin x)\,dx$
2. $\int \left(x + \frac{1}{\sqrt{1-x^2}}\right) dx$
3. $\int (x^3 + \cos x)\,dx$
4. $\int (8t^4 - 5\cos t)\,dt$
5. $\int \left(e^x - x^2 - \frac{1}{x} + \cos x\right) dx$
6. $\int \left(3^x - \frac{3}{x} + \cos x\right) dx$
7. $\int (e^\theta + \theta^2)\,d\theta$
8. $\int \frac{\sqrt[3]{x^2} - x^{5/2}}{\sqrt{x}}\,dx$
9. $\int x^2 \sin x^3\,dx$
10. $\int \tan x \sec^2 x\,dx$
11. $\int x^2 e^{(x^3)}\,dx$
12. $\int x e^{(x^2)}\,dx$
13. $\int (x+2)^5\,dx$
14. $\int \frac{dx}{3x+4}$
15. $\int x^2 e^{(4x^3)}\,dx$
16. $\int (1 + 3x^2)\exp(x + x^3)\,dx$
17. $\int 2\cos^2 2x \sin 2x\,dx$
18. $\int 3\sin 3x \cos 3x\,dx$
19. $\int x \tan^{-1} x\,dx$
20. $\int x\sqrt{5 - x^2}\,dx$
21. $\int \left[\frac{1}{\sqrt{4 - t^2}} + t^2\right] dt$
22. $\int \frac{e^{2x}}{1 + e^{4x}}\,dx$
23. $\int x e^{4x}\,dx$
24. $\int x e^{6x}\,dx$
25. $\int x^2 \cos x\,dx$
26. $\int x^2 e^{2x}\,dx$
27. $\int e^{-x} \cos x\,dx$
28. $\int e^{2x} \tan e^{2x}\,dx$
29. $\int x^2 \ln 3x\,dx$
30. $\int x^3 \ln x\,dx$
31. $\int x\sqrt{x+3}\,dx$
32. $\int x^2\sqrt{x+1}\,dx$
33. $\int x \cos 3x\,dx$
34. $\int t \cos 2t\,dt$
35. $\int 3x \cos 2x\,dx$
36. $\int \sin 2x \cos x\,dx$
37. $\int x^3 e^{(x^2)}\,dx$
38. $\int x^5 e^{(x^3)}\,dx$
39. $\int x(\ln x)^2\,dx$
40. $\int (\ln x)^2\,dx$
41. $\int e^{\sqrt{x}}\,dx$
42. $\int \frac{dx}{x^2 + 2x + 3}$ (Complete the square.)
43. $\int [\cos x]\ln(\sin x)\,dx$
44. $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$
45. $\int \tan^{-1} x\,dx$
46. $\int \cos^{-1}(12x)\,dx$

Evaluate the definite integrals in Exercises 47–58.

47. $\int_{-1}^{0} x e^{-x}\,dx$
48. $\int_1^e x \ln(5x)\,dx$
49. $\int_0^{\pi/5} x \sin 5x\,dx$
50. $\int_0^{\pi/4} x \cos 2x\,dx$
51. $\int_1^2 x^{-2}\cos(1/x)\,dx$
52. $\int_0^{\pi/2} x^2 \cos(x^3)\sin(x^3)\,dx$
53. $\int_0^{\pi/4} x \tan^{-1} x\,dx$
54. $\int_1^{\ln(\pi/4)} e^x \tan e^x\,dx$
55. $\int_{a+1}^{a+2} \frac{t}{\sqrt{t-a}}\,dt$ (substitute $x = \sqrt{t-a}$)
56. $\int_0^1 \frac{\sqrt{x}}{x+1}\,dx$
57. $\int_0^1 x\sqrt{2x+3}\,dx$
58. $\int_0^{\sqrt{3}} \frac{3}{3+u^2}\,du$

In Exercises 59–66, sketch the region under the graph the given function on the given interval and find its area.

59. $40 - x^3$ on $[0, 3]$
60. $\sin x + 2x$ on$[0, 4\pi]$

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61. $3x/\sqrt{x^2+9}$ on $[0,4]$
62. $x\sin^{-1}x+2$ on $[0,1]$
63. $\sin x$ on $[0,\pi/4]$
64. $\sin 2x$ on $[0,\pi/2]$
65. $1/x$ on $[2,4]$
66. xe^{-2x} on $[0,1]$

67. Let R_n be the region bounded by the x axis, the line $x=1$, and the curve $y=x^n$. The area of R_n is what fraction of the area of the triangle R_1?
68. Find the area under the graph of $f(x)=x/\sqrt{x^2+2}$ from $x=0$ to $x=2$.
69. Find the area between the graphs of $y=-x^3-2x-6$ and $y=e^x+\cos x$ from $x=0$ to $x=\pi/2$.
70. Find the area above the n^{th} bend of $y=x\sin x$ which liesbelow the x axis. (See Fig. 7.4.2).
71. Water is flowing into a tank with a rate of $10(t^2+\sin t)$ liters per minute after time t. Calculate: (a) the number of liters stored after 30 minutes, starting at $t=0$; (b) the average flow rate in liters per minute over this 30-minute interval.
72. The velocity of a train fluctuates according to the formula $v=(100+e^{-3t}\sin 2\pi t)$ kilometers per hour. How far does the train travel: (a) between $t=0$ and $t=1$?; (b) between $t=100$ and $t=101$?
73. Evaluate $\int \sin(\pi x/2)\cos(\pi x)\,dx$ by integrating by parts two different ways and comparing the results.
74. Do Exercise 73 using the product formulas for sine and cosine.
75. Evaluate $\int \sqrt{(1+x)/(1-x)}\,dx$. [*Hint*: Multiply numerator and denominator by $\sqrt{1+x}$.]
76. Substitute $x=\sin u$ to evaluate

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

and

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}}; \qquad 0<x<1.$$

77. Evaluate:

(a) $\displaystyle\int \frac{\ln x}{x}\,dx,$

(b) $\displaystyle\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}}$, (use $x=3\tan u$).

78. (a) Prove the following reduction formula:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int \sin^{n-2}x\,dx$$

if $n\geqslant 2$, by integration by parts, with $u=\sin^{n-1}x$, $v=-\cos x$.

(b) Evaluate $\int \sin^2 x\,dx$ by using this formula.

(c) Evaluate $\int \sin^4 x\,dx$.

79. Find $\int x^n \ln x\,dx$ using $\ln x=(1/(n+1))\ln x^{n+1}$ and the substitution $u=x^{n+1}$.
80. (a) Show that:

$$\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1}-\frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$$

(b) Evaluate $\int_1^2 x^2(\ln x)^2\,dx$.

81. The charge Q in coulombs for an RC circuit with sinusoidal switching satisfies the equation

$$\frac{dQ}{dt}+\frac{1}{0.04}Q=100\sin\left(\frac{\pi}{2-5t}\right),\quad Q(0)=0.$$

The solution is

$$Q(t)=100e^{-25t}\int_0^t e^{25x}\cos 5x\,dx.$$

(a) Find Q explicitly by means of integration by parts.

(b) Verify that $Q(1.01)=0.548$ coulomb. [*Hint*: Be sure to use radians throughout the calculation.]

82. What happens if $\int f(x)\,dx$ is integrated by parts with $u=f(x)$, $v=x$?

★83. Arthur Perverse believes that the product rule for integrals ought to be that $\int f(x)g(x)\,dx$ equals $f(x)\int g(x)\,dx+g(x)\int f(x)\,dx$. We wish to show him that this is not a good rule.

(a) Show that if the functions $f(x)=x^m$ and $g(x)=x^n$ satisfy Perverse's rule, then for fixed n the number m must satisfy a certain quadratic equation (assume $n,m\geqslant 0$).

(b) Show that the quadratic equation of part (a) does not have any real roots for any $n\geqslant 0$.

(c) Are there *any* pairs of functions, f and g, which satisfy Perverse's rule? (Don't count the case where one function is zero.)

★84. Derive an integration formula obtained by reading the quotient rule for derivatives backwards.

★85. Find $\int xe^{ax}\cos(bx)\,dx$.

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